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On the Bourgain, Brezis, and Mironescu Theorem Concerning Limiting Embeddings of Fractional Sobolev Spaces

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The article is concerned with the Bourgain, Brezis and Mironescu theorem on the asymptotic behaviour of the norm of the Sobolev-type embedding operator: $\mathscr{W}^{s,p} \to L^{pn/(n-sp)}$ as $s \uparrow 1$ and $s \uparrow n/p$. Their result is extended to all values of $s \in (0,1)$ and is supplied with an elementary proof. The relation

$$\lim_{s \downarrow 0} s \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy = 2p^{-1}|S^{n-1}|\, \|u\|^p_{L^p(\mathbf{R}^n)}$$

is proved.

1. INTRODUCTION

Let $s \in (0,1)$ and let $p \geqslant 1$. We introduce the space $\mathscr{W}^{s,p}_0(\mathbf{R}^n)$ as the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$\left(\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \right)^{1/p}.$$

We also need the space $\mathscr{W}^{s,p}_\perp(Q)$ of functions defined on the cube $Q = \{x \in \mathbf{R}^n : |x_i| < 1/2, 1 \leqslant i \leqslant n\}$ which are orthogonal to 1 and have the finite norm

$$\left(\int_Q \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy \right)^{1/p}.$$

The main result of the recent paper by Bourgain *et al.* [BBM$_1$] is the inequality

$$\|u\|^p_{L^q(Q)} \leqslant c(n) \frac{1-s}{(n-sp)^{p-1}} \|u\|^p_{\mathscr{W}^{s,p}_\perp(Q)}, \tag{1}$$

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where $u \in \mathscr{W}_{\perp}^{s,p}(Q)$, $1/2 \leqslant s < 1$, $sp < n$, $q = pn/(n - sp)$, and $c(n)$ depends only on n.

The present article is a direct outgrowth of this result. Figuring out a similar estimate for functions in $\mathscr{W}_0^{s,p}(\mathbf{R}^n)$, valid for the whole interval $0 < s < 1$, one could anticipate the appearance of the factor $s(1-s)$ in the right-hand side, since the norm in $\mathscr{W}_0^{s,p}(\mathbf{R}^n)$ blows up both as $s \uparrow 1$ and $s \downarrow 0$. The following theorem shows that this is really the case.

THEOREM 1. *Let $n \geqslant 1$, $p \geqslant 1$, $0 < s < 1$, and $sp < n$. Then, for an arbitrary function $u \in \mathscr{W}_0^{s,p}(\mathbf{R}^n)$, there holds*

$$\|u\|_{L^q(\mathbf{R}^n)}^p \leqslant c(n,p) \frac{s(1-s)}{(n-sp)^{p-1}} \|u\|_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)}^p, \tag{2}$$

where $q = pn/(n-sp)$ and $c(n,p)$ is a function of n and p.

From Theorem 1, one can derive inequality (1) for all $s \in (0,1)$ with a constant c depending both on n and p (Corollary 2). In the case $s \geqslant 1/2$ considered in [BBM$_1$], one has $1 < p < 2n$ and therefore the dependence of the constant c on p can be eliminated. Thus, we arrive at the Bourgain–Brezis–Mironescu result and extend it to the values $s < 1/2$.

The proof given in [BBM$_1$] relies upon some advanced harmonic analysis and is quite complicated. Our proof of (2) is straightforward and rather simple. It is based upon an estimate of the best constant in a Hardy-type inequality for the norm in $\mathscr{W}_0^{s,p}(\mathbf{R}^n)$, which is obtained in Theorem 2 and is of independent interest.

In Theorem 3 we derive a formula for $\lim_{s \downarrow 0} s\|u\|_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)}^p$ which complements an analogous formula for $\lim_{s \uparrow 1}(1-s)\|u\|_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)}^p$ found in [BBM$_2$].

2. HARDY-TYPE INEQUALITIES

THEOREM 2. *Let $n \geqslant 1$, $p \geqslant 1$, $0 < s < 1$, and $sp < n$. Then, for an arbitrary function $u \in \mathscr{W}_0^{s,p}(\mathbf{R}^n)$, there holds*

$$\int_{\mathbf{R}^n} |u(x)|^p \frac{dx}{|x|^{sp}} \leqslant c(n,p) \frac{s(1-s)}{(n-sp)^p} \|u\|_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)}^p. \tag{3}$$

Proof. Let

$$\psi(h) = |S^{n-1}|^{-1} n(n+1)(1-|h|)_+,$$

where $h \in \mathbf{R}^n$ and plus stands for the nonnegative part of a real-valued function. We introduce the standard extension of u onto $\mathbf{R}^{n+1}_+ = \{(x,z) : x \in \mathbf{R}^n, z > 0\}$

$$U(x,z) := \int_{\mathbf{R}^n} \psi(h) u(x+zh)\, dh.$$

A routine majoration implies

$$|\nabla U(x,z)| \leqslant \frac{n(n+1)(n+2)}{z|S^{n-1}|} \int_{|h|<1} |u(x+zh) - u(x)|\, dh.$$

Hence and by Hölder's inequality one has

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p \, dx\, dz \\ &\quad \leqslant \frac{n}{|S^{n-1}|}(n+1)^p (n+2)^p \\ &\quad \times \int_0^\infty z^{-1-ps} \int_{|h|<1} \int_{\mathbf{R}^n} |u(x+zh) - u(x)|^p \, dx\, dh\, dz. \end{aligned} \tag{4}$$

Setting $\eta = zh$ and changing the order of integration, one can rewrite (4) as

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)} |\nabla U(x,z)|^p \, dx\, dz \\ &\quad \leqslant \frac{n(n+1)^p (n+2)^p}{|S^{n-1}|(sp+n)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy. \end{aligned} \tag{5}$$

By Hardy's inequality,

$$\int_0^{|x|} z^{-1-sp} \left| \int_0^z \varphi(\tau)\, d\tau \right|^p dz \leqslant s^{-p} \int_0^{|x|} z^{-1+p(1-s)} |\varphi(z)|^p \, dz$$

one has

$$\begin{aligned} \frac{|u(x)|^p}{|x|^{sp}} &= p(1-s) \int_0^{|x|} z^{-1+p(1-s)}\, dz\, \frac{|u(x)|^p}{|x|^p} \\ &\leqslant p(1-s) \int_0^{|x|} z^{-1-sp}\, dz \left(\int_0^z \left(\left| \frac{\partial U}{\partial \tau}(x,\tau) \right| + \frac{|U(x,\tau)|}{|x|} \right) d\tau \right)^p \\ &\leqslant \frac{p(1-s)}{s^p} \int_0^{|x|} z^{-1+p(1-s)} \left(\left| \frac{\partial U}{\partial z}(x,z) \right| + \frac{|U(x,z)|}{|x|} \right)^p dz. \end{aligned}$$

Now, the integration over $\mathbf{R}^n$ and Minkowski's inequality imply

$$\int_{\mathbf{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\,dx \leqslant \frac{p(1-s)}{s^p}\left(\left(\int_{\mathbf{R}^n}\int_0^\infty z^{-1+p(1-s)}\left|\frac{\partial U}{\partial z}(x,z)\right|^p dz\,dx\right)^{1/p} + A\right)^p, \quad (6)$$

where

$$A := \left(\int_{\mathbf{R}^n}\int_0^{|x|} z^{-1+p(1-s)}|x|^{-p}|U(x,z)|^p\,dz\,dx\right)^{1/p}.$$

Clearly,

$$A^p \leqslant 2^{p/2}\int_{\mathbf{R}^n} dx \int_0^\infty z^{-1+p(1-s)}\frac{|U(x,z)|^p}{(x^2+z^2)^{p/2}}\,dz\,dx,$$

which does not exceed

$$2^{p/2}\int_{S^n_+}(\cos\theta)^{-1+p(1-s)}\int_0^\infty |U|^p\rho^{n-1-sp}\,d\rho\,d\sigma, \quad (7)$$

where $\rho = (x^2+z^2)^{1/2}$, $\cos\theta = z/\rho$, $d\sigma$ is an element of the surface area on the unit sphere S^n, and S^n_+ is the upper half of S^n. Using Hardy's inequality

$$\int_0^\infty |U|^p\rho^{n-1-sp}\,d\rho \leqslant \left(\frac{p}{n-sp}\right)^p \int_0^\infty \left|\frac{\partial U}{\partial \rho}\right|^p \rho^{n-1+p(1-s)}\,d\rho,$$

one arrives at the estimate

$$A^p \leqslant \left(\frac{2^{1/2}p}{n-sp}\right)^p \int_0^\infty \int_{\mathbf{R}^n} z^{-1+p(1-s)}|\nabla U(x,z)|^p\,dx\,dz.$$

Combining this with (6), one obtains

$$\int_{\mathbf{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,dx \leqslant \frac{p(1-s)}{s^p}\left(1+\frac{2^{1/2}p}{n-sp}\right)^p\int_0^\infty\int_{\mathbf{R}^n} z^{-1+p(1-s)}|\nabla U(x,z)|^p\,dx\,dz$$

which, along with (5), gives

$$\int_{\mathbf{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,dx \leqslant \frac{(1-s)}{(n-sp)^p}\frac{p(n+2p)^{3p}}{|S^{n-1}|s^p}\,\|u\|^p_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)}. \quad (8)$$

In order to justify (3) we need to improve (8) for small values of s. Clearly,

$$\frac{|S^{n-1}|}{2^{sp}sp}\int_{\mathbf{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,dx = \int_{\mathbf{R}^n}\int_{|x-y|>2|x|}\frac{dy}{|x-y|^{n+sp}}\,|u(x)|^p\,dx.$$

Since $|x-y|>2|x|$ implies $2|y|/3<|x-y|<2|y|$, we obtain

$$\left(\frac{|S^{n-1}|}{2^{sp}sp}\int_{\mathbf{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,dx\right)^{1/p}\leqslant\left(\int_{\mathbf{R}^n}\int_{|x-y|>|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{1/p}$$

$$+\left(|S^{n-1}|\frac{3^{sp}-1}{2^{sp}sp}\int_{\mathbf{R}^n}\frac{|u(y)|^p}{|y|^{sp}}\,dy\right)^{1/p}.$$

Hence,

$$\left(\frac{|S^{n-1}|}{2^{sp}sp}\right)^{1/p}(1-(3^{sp}-1)^{1/p})\left(\int_{\mathbf{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,dx\right)^{1/p}\leqslant 2^{-1/p}||u||_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)}.$$

Let δ be an arbitrary number in $(0,1)$. If $s\leqslant(4p)^{-1}\delta^p$, we conclude

$$\int_{\mathbf{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,dx\leqslant\frac{2^{sp-1}sp}{|S^{n-1}|(1-\delta)^p}\,||u||^p_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)}. \tag{9}$$

Setting $\delta=2^{-1}$ and comparing this inequality with (8), we arrive at (3) with $c(n,p)=|S^{n-1}|^{-1}(n+2p)^{3p}p^{p+2}2^{(n+1)(n+2)}$. The proof is complete. ∎

From Theorem 2, we shall deduce an inequality, analogous to (3), for functions defined on the cube Q. Unlike (3), this inequality contains no factor s in the right-hand side, which is not surprising, because, for smooth u, the norm $||u||_{\mathscr{W}_\perp^{s,p}(Q)}$ tends to a finite limit as $s\downarrow 0$.

Corollary 1. *Let $n\geqslant 1$, $p\geqslant 1$, $0<s<1$, and $sp<n$. Then any function $u\in\mathscr{W}_\perp^{s,p}(Q)$ satisfies*

$$\int_Q |u(x)|^p\frac{dx}{|x|^{sp}}\leqslant c(n,p)\frac{1-s}{(n-sp)^p}\,||u||^p_{\mathscr{W}_\perp^{s,p}(Q)}. \tag{10}$$

Proof. Let us preserve the notation u for the mirror extension of $u\in\mathscr{W}_\perp^{s,p}(Q)$ to the cube $3Q$, where aQ stands for the cube obtained from Q by dilation with the coefficient a. We choose a cut-off function η, equal to 1 on Q and vanishing outside $2Q$, say, $\eta(x)=\prod_{i=1}^n\min\{1,2(1-x_i)_+\}$. By Theorem 2, it is enough to prove that

$$||\eta u||^p_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)}\leqslant s^{-1}c(n,p)||u||^p_{\mathscr{W}_\perp^{s,p}(Q)}. \tag{11}$$

Clearly, the norm in the left-hand side is majorized by

$$\left(\int_{3Q}\int_{3Q}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,\eta(y)^p\,dy\right)^{1/p} + \left(\int_{3Q}\int_{3Q}\frac{|\eta(x)-\eta(y)|^p}{|x-y|^{n+sp}}\,dx|u(y)|^p\,dy\right)^{1/p} + \left(2\int_{3Q}\int_{\mathbf{R}^n\setminus 3Q}\frac{dy}{|x-y|^{n+sp}}\,|(\eta u)(x)|^p\,dx\right)^{1/p}.$$

The first term does not exceed $6^{n/p}\|u\|_{\mathscr{W}^{s,p}_{\perp}(Q)}$; the second term is not greater than

$$2n^{1/2}\left(\int_{3Q}\int_{3Q}\frac{dy}{|x-y|^{n-p(1-s)}}|u(y)|^p\,dy\right)^{1/p} \leqslant n3^{2+n/p}\left(\frac{|S^{n-1}|}{p(1-s)}\right)^{1/p}\|u\|_{L^p(Q)},$$

and the third one is dominated by

$$\left(2\int_{2Q}\int_{|x-y|>1/2}\frac{dy}{|x-y|^{n+sp}}\,|u(x)|^p\,dx\right)^{1/p} \leqslant \left(\frac{2^{n+1+p}}{sp}\right)^{1/p}\|u\|_{L^p(Q)}.$$

Summing up these estimates, one obtains

$$\|\eta u\|_{\mathscr{W}^{s,p}_{0}(\mathbf{R}^n)} \leqslant 6^{n/p}\|u\|_{\mathscr{W}^{s,p}_{0}(Q)} + n3^{2+n/p}p^{-1/p}(s^{-1/p}+(1-s)^{-1/p})\|u\|_{L^p(Q)}. \tag{12}$$

Recalling that $u \perp 1$ on Q, one has for any $z \in Q$

$$\int_Q |u(x)|^p\,dx \leqslant \int_Q\int_Q |u(x)-u(y)|^p\,dx\,dy \leqslant 2^p \int_Q |u(x)-u(z)|^p\,dx.$$

Hence and by the obvious inequality

$$\int_{2Q}\frac{dz}{|x-z|^{n-p(1-s)}} > \int_{|z-x|<1/2}\frac{dz}{|x-z|^{n-p(1-s)}} = \frac{|S^{n-1}|}{p(1-s)2^{p(1-s)}},$$

where $x \in Q$, it follows that

$$\int_Q |u(x)|^p\,dx \leqslant \frac{2^{p(2-s)}p(1-s)}{|S^{n-1}|}\int_{2Q}\int_Q \frac{|u(x)-u(z)|^p}{|x-z|^{n-p(1-s)}}\,dx\,dz.$$

Thus,

$$\|u\|_{L^p(Q)} \leqslant 2^{2+n/p}n^{1/2}\left(\frac{p(1-s)}{|S^{n-1}|}\right)^{1/p}\|u\|_{\mathscr{W}^{s,p}_{\perp}(Q)}.$$

Combining this inequality with (12), we justify (11) and hence complete the proof. ∎

3. SOBOLEV EMBEDDINGS

Proof of Theorem 1. It is well known that the fractional Sobolev norm of order $s \in (0,1)$ is nonincreasing with respect to symmetric rearrangement of functions decaying to zero at infinity (see [W, AL, Theorem 9.2; Ci]). Let $v(|x|)$ denote the rearrangement of $|u(x)|$. Then

$$\|u\|_{L^q(\mathbf{R}^n)} = \left(\frac{|S^{n-1}|}{n} \int_0^\infty v(r)^q \, d(r^n) \right)^{1/q}, \tag{13}$$

where $|S^{n-1}|$ is the area of the unit sphere S^{n-1}. Recalling that an arbitrary non-negative nonincreasing function f on the semi-axis $(0,\infty)$ satisfies

$$\int_0^\infty f(t)^\lambda \, d(t^\lambda) \leqslant \lambda \int_0^\infty \left(\int_0^t f(\tau)\, d\tau \right)^{\lambda-1} f(t)\, dt = \left(\int_0^\infty f(t)\, dt \right)^\lambda, \quad \lambda \geqslant 1$$

(see [HLP]), one finds that the right-hand side in (13) does not exceed

$$\left(\frac{|S^{n-1}|}{n} \right)^{1/q} \left(\int_0^\infty v(r)^p \, d(r^{n-sp}) \right)^{1/p} = \frac{(n-sp)^{1/p}}{n^{1/q} |S^{n-1}|^{s/n}} \left(\int_{\mathbf{R}^n} v(|x|)^p \frac{dx}{|x|^{sp}} \right)^{1/p}.$$

We now see that (2) results from inequality (3). ∎

COROLLARY 2. *Let $n \geqslant 1$, $p \geqslant 1$, $0 < s < 1$, and $sp < n$. Then any function $u \in \mathscr{W}_\perp^{s,p}(Q)$ satisfies*

$$\|u\|^p_{L^p(Q)} \leqslant c(n,p) \frac{1-s}{(n-sp)^{p-1}} \|u\|^p_{\mathscr{W}_\perp^{s,p}(Q)}.$$

Proof. Let η be the same cut-off function as in Corollary 1. The result follows by combining inequality (11) with Theorem 1 where u is replaced by ηu. ∎

4. ASYMPTOTICS OF THE NORM IN $\mathscr{W}_0^{s,p}(\mathbf{R}^n)$ AS $s \downarrow 0$

THEOREM 3. *For any function $u \in \bigcup_{0<s<1} \mathscr{W}_0^{s,p}(\mathbf{R}^n)$, there exists the limit*

$$\lim_{s \downarrow 0} s \|u\|^p_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)} = 2p^{-1} |S^{n-1}| \, \|u\|^p_{L^p(\mathbf{R}^n)}.$$

Proof. Since δ can be chosen arbitrarily small, inequality (9) implies

$$\liminf_{s\downarrow 0} s\|u\|^p_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)} \geqslant 2p^{-1}|S^{n-1}|\,\|u\|^p_{L^p(\mathbf{R}^n)}. \tag{14}$$

Let us majorize the upper limit. By (14), it suffices to assume that $u \in L^p(\mathbf{R}^n)$. Clearly,

$$\begin{aligned} s\|u\|^p_{\mathscr{W}_0^{s,p}(\mathbf{R}^n)} \leqslant 2\Bigg\{ &\left(s\int_{\mathbf{R}^n}\int_{|y|\geqslant 2|x|} \frac{dy}{|x-y|^{n+sp}}|u(x)|^p\,dx\right)^{1/p} \\ &+\left(s\int_{\mathbf{R}^n}|u(y)|^p\int_{|y|\geqslant 2|x|}\frac{dx\,dy}{|x-y|^{n+sp}}\right)^{1/p}\Bigg\}^p \\ &+\ 2s\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy. \end{aligned}$$

The first term in braces does not exceed

$$\left(s\int_{\mathbf{R}^n}\int_{|y|\geqslant |x|}\frac{dy}{|x-y|^{n+sp}}|u(x)|^p\,dx\right)^{1/p} = \frac{|S^{n-1}|^{1/p}}{p^{1/p}}\left(\int_{\mathbf{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,dx\right)^{1/p}$$

hence its $\limsup_{s\downarrow 0}$ is dominated by $|S^{n-1}|^{1/p}p^{-1/p}\|u\|_{L^p(\mathbf{R}^n)}$. The second term in braces is not greater than

$$s^{1/p}\left(2^{n+sp}\int_{\mathbf{R}^n}\frac{|u(y)|^p}{|y|^{n+sp}}\,dy\int_{|x|<|y|/2}dx\right)^{1/p} = 2^s\left(\frac{s}{p}|S^{n-1}|\right)^{1/p}\left(\int_{\mathbf{R}^n}\frac{|u(y)|^p}{|y|^{sp}}\,dy\right)^{1/p},$$

so it tends to zero as $s\downarrow 0$.

We claim that

$$\limsup_{s\downarrow 0} s\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = 0. \tag{15}$$

By assumption of the theorem, $u \in \mathscr{W}_0^{\tau,p}(\mathbf{R}^n)$ for a certain $\tau \in (0,1)$. Let N be an arbitrary number greater than 1 and let $s<\tau$. We have

$$\begin{aligned} &2s\int_{\mathbf{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\ &\quad\leqslant 2sN^{p(\tau-s)}\int_{\mathbf{R}^n}\int_{\substack{|x|<|y|<2|x|\\ |x-y|\leqslant N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+\tau p}}\,dx\,dy \\ &\quad\quad+\ 2s\int_{\mathbf{R}^n}\int_{\substack{|x|<|y|<2|x|\\ |x-y|>N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy. \end{aligned}$$

The first term in the right-hand side tends to zero as $s \downarrow 0$ and the second one does not exceed

$$2^{p+1} s \int_{|x|>N/3} \int_{|x-y|>N} \frac{dy}{|x-y|^{n+sp}} |u(x)|^p \, dx \leqslant c(n,p) \int_{|x|>N/3} |u(x)|^p \, dx,$$

which is arbitrarily small if N is sufficiently large. The proof is complete. ∎

Remark. Since the proof of Theorem 3 holds for vector-valued functions, one can write

$$\lim_{s \downarrow 0} s \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|\nabla u(x) - \nabla u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy = 2p^{-1} |S^{n-1}| \int_{\mathbf{R}^n} |\nabla u(x)|^p \, dx$$

for any function u such that $\nabla u \in \bigcup_{0<s<1} \mathscr{W}_0^{s,p}(\mathbf{R}^n)$. This formula complements the following relation which was established in [BBM$_2$]:

$$\lim_{s \uparrow 1} (1-s) \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy = \int_{S^{n-1}} |\cos \theta|^p \, d\sigma \int_{\mathbf{R}^n} |\nabla u(x)|^p \, dx,$$

where θ is the angle deviation from the vertical.

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